Stochastic fields on a lattice

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A Caldeira-Leggett-type system-bath Hamiltonian is used to construct a new class of local stochastic equations on a lattice and to determine the conditions under which the lattice field evolves to thermal equilibrium. Both scalar and two-dimensional vector fields driven by multiplicative noise are considered. The latter model is developed to describe magnetization dynamics with spatial dispersion of relaxation. A systematic method of constructing stochastic field equations from a complex dispersion relation is proposed. The corresponding lattice Fokker-Planck equation is written down and it is shown that thermal equilibrium is its stationary state.

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Nondissipative dynamics of magnetization are described by the equation $d\mathbf{M}/dt=\gamma_0\mathbf{M}\times(-\delta H/\delta\mathbf{M})$, where γ_0 is the gyromagnetic ratio and H is the total free energy. It is a function of the magnetization components M_i (homogeneous rotation of the vector $\mathbf{M}$) and the spatial derivatives $\partial M_i/\partial x_j$ (exchange interactions). The equation is well known for its soliton solutions [1] representing domain walls; the question is how to introduce into it dissipative forces. In 1955 Gilbert [2] generalized it as $d\mathbf{M}/dt=\gamma_0\mathbf{M}\times(-\delta H/\delta\mathbf{M}-\epsilon_g d\mathbf{M}/dt)$, where ϵ_g is a dissipative constant associated with homogeneous rotation (rotation in unison). The equation conserves $|\mathbf{M}|$ and has been used to study both coherent rotation in single domain particles [2,3] and dynamics of domain walls [1,4]. In 1984, however, Bar'yakhtar [5] pointed out that the dissipation constant deduced experimentally from domain-wall drag is much larger than the one deduced from ferromagnetic resonance. He concluded that there must exist at least one more dissipative term in the phenomenological equation of motion and that this term must be associated with propagation of spatial inhomogeneities. To find this term Bar'yakhtar had to assume that the total magnetic moment of the sample, $\mathcal{M}_{\rm tot}=\int dV\,\mathbf{M}$, is conserved rather than the local magnetic moment density $|\mathbf{M}(t,\mathbf{r})|$, and much of the subsequent work [6] was then devoted to showing that the nonconservation of $|\mathbf{M}|$ is small.

The motivation of this article is the need to derive a stochastic equation of motion which takes into account both nondispersive (spatially homogeneous) and dispersive effects of dissipation, conserves $|\mathbf{M}|$ locally, and, at the same time, evolves towards thermal equilibrium. Our starting point is a system-bath Hamiltonian which follows closely (but in real time) the celebrated model of Caldeira and Leggett [7]. The method is eminently well suited to the derivation of stochastic equations of motion together with the associated fluctuation-dissipation (FD) theorems and has been used previously [3] to derive a stochastic generalization of Gilbert's equation for rotation in unison with damping of specific point symmetry. In this instance it was found that in systems driven by multiplicative vector noise coupled to both generalized position and momentum the FD theorem is a necessary, but not sufficient, condition for the existence of thermal equilibrium.

Here we develop a stochastic theory on lattice, derive a Fokker-Planck equation for a countable number of degrees of freedom, and write down the final results in continuum limit. Because of the complications associated with vector field theory we consider first the scalar case and derive a class of stochastic field equations with additive and multiplicative scalar noise field. Such equations may be used to describe, e.g., a damped Josephson junction [8]. The more complicated magnetization dynamics driven by a multiplicative vector noise are treated next: Rotation of the vector $\mathbf{M}$ has two degrees of freedom and a stochastic description of the two-dimensional field $\mathbf{M}(t,\mathbf{r})$ over a three-dimensional space is required if spatial dispersion of magnetization is to be taken into account also. Only an approximate Fokker-Planck equation valid in the underdamped limit can be written down in this case.

The dissipative coupling of a Josephson junction to a heat bath was described by Caldeira and Leggett [7] via the bilinear interaction Hamiltonian $H_{\rm int}=\sum_\alpha c_\alpha q_\alpha Q$, where Q is a junction variable coupled to a linear superposition of the bath variables q_α. We generalize this to a form invariant under both space and time inversion:

$$H_{\rm int}=\sum_\alpha[c_\alpha(\mathbf{r})q_\alpha+\tilde{d}_\alpha(\mathbf{r})(\nabla q_\alpha)\cdot\nabla]Q\ . \qquad (1)$$

Here $\mathbf{r}$ is spatial coordinate and $\nabla=\partial/\partial\mathbf{r}$. We further introduce a lattice with lattice constant a and write (in one spatial dimension for brevity) at the point $r=ia$

$$\frac{\partial\xi}{\partial r}=(2a)^{-1}\{\xi[a(i+1)]-\xi[a(i-1)]\}$$
$$\equiv(2a)^{-1}\{\xi(i+1)-\xi(i-1)\}\ . \qquad (2)$$

Here ξ is an arbitrary functional in the bath or system variables [e.g., $\xi=q_\alpha(i)$ or $\xi=\partial/\partial Q(i)$] and the second equality defines abbreviated notation on the lattice; for the system variables we shall also write $Q(i)\equiv Q_i$. In these lattice variables the total system-bath Hamiltonian including the counterterm [3,7] becomes, in one dimension,

$$H_{\rm tot}=\sum_i \left\{ H[Q_i,P_i]+\frac{1}{2}\sum_\alpha \{p_\alpha^2(i)+\omega_\alpha^2\bar{q}_\alpha^2(i)\} \right\} , \tag{3}$$

where

$$\bar{q}_\alpha(i)=q_\alpha(i)+\omega_\alpha^{-2}\{c_\alpha(i)Q_i+d_\alpha(i-1)[Q_i-Q_{i-2}] -d_\alpha(i+1)[Q_{i+2}-Q_i]\} \tag{4}$$

and $d_\alpha(i)=(2a)^{-2}\tilde{d}_\alpha(r)|_{r=ia}$. The pairs $\{Q_i,P_i\}$ and $\{q_\alpha(i),p_\alpha(i)\}$ at the point $r=ia$ are the conjugate variables of the system and the bath, respectively. Spatial inhomogeneities within the bath arise in the Hamiltonian (3) only via interactions with the spatially dispersive system. It is also possible to add an elastic bath term into $H_{\rm tot}$, e.g., $H_{\rm el}=\sum_\alpha e_\alpha(\mathbf{r})[(\nabla q_\alpha)^2+\Omega_\alpha^2(\nabla p_\alpha)^2]$, and impose some boundary conditions on $\{q_\alpha,p_\alpha\}$, but this leads to a complicated nonlocal theory characterized by spatial correlations of bath variables and dependent on their boundary values. Our aim here is a local theory and this venue shall not be pursued further.

Derivation of stochastic equations of motion from Eqs. (3) and (4) follows a standard procedure [3]: The solution of the inhomogeneous linear Hamilton's equations $\dot{q}_\alpha(i)=\partial H_{\rm tot}/\partial p_\alpha(i)$, $\dot{p}_\alpha(i)=-\partial H_{\rm tot}/\partial q_\alpha(i)$ is substituted into the equations $\dot{Q}_i=\partial H_{\rm tot}/\partial P_i$, $\dot{P}_i=-\partial H_{\rm tot}/\partial Q_i$ and the dissipative kernel $\hat{\gamma}(z)$ is constructed [3,7] from the coupling coefficients $c_\alpha(i)$, $d_\alpha(i)$. The noise term then originates from the homogeneous solution of the bath equations. Let it be designated $b(t,i)$. The construction guarantees that after averaging over initial states of the bath the relation $\mathcal{L}\langle b(t,i)b(t',j)\rangle=T\hat{\gamma}(z)\delta_{i,j}$ holds, $\mathcal{L}$ being a Laplace transform operator and T temperature. This is a necessary [but not always sufficient; see the discussion of Eqs. (19) and (20) below] condition for the existence of thermal equilibrium. In the Markovian limit the noise $b(t,i)$ becomes the Wiener process $w(t,i)$ and the resultant stochastic equations of motion can symbolically be written as

$$\dot{P}=-H_Q-\Delta_{\epsilon,\eta}^2\dot{Q}-(2T)^{1/2}\Delta_{\epsilon,\eta}\dot{w} , \tag{5}$$

$$\dot{Q}=H_P , \tag{6}$$

where $H_Q=\delta H/\delta Q\rightarrow\partial H/\partial Q_i=H_{Q_i}$ in the lattice equation for $\dot{P}_i$, etc. We introduced here the dissipation operator

$$\Delta_{\epsilon,\eta}=\epsilon-\nabla\cdot(\eta\nabla) , \tag{7}$$

in which ϵ and η are in general functions of $\mathbf{r}$ and the action of ∇ on the lattice is defined by Eq. (2). Generalization to system with memory is trivial. In Eqs. (5) and (6) the operator $\Delta_{\epsilon,\eta}$ formally represents a dissipation constant. Their noiseless part is well defined also in the continuum limit while the noise term makes sense only in terms of lattice differences since the Wiener processes on neighboring nodes are uncorrelated by assumption: $\Delta_{\epsilon,\eta}\dot{w}=(\epsilon+2\tilde{\eta})\dot{w}_i-\tilde{\eta}(\dot{w}_{i+2}+\dot{w}_{i-2})$ for position independent ϵ and $\eta=4a^2\tilde{\eta}$. For the quadratic Hamiltonian density $\hat{H}=\frac{1}{2}[P^2+\Omega^2Q^2+\Xi^2(\nabla Q)^2]$ the noiseless equations (5) and (6) have a running wave solution with the complex dispersion relation

$$\omega=(\omega_1^2-\omega_2^2)^{1/2}-i\omega_2 , \tag{8}$$

where $\omega_1^2=\Omega^2+\Xi^2k^2$, $\omega_2=(\epsilon+\eta k^2)^2/2$ and $k=|\mathbf{k}|$. As expected, waves with large $\mathbf{k}$ vector are attenuated preferentially.

Given the set (5) and (6) it is possible to find [9] the (Stratonovich) Fokker-Planck operator L for the joint probability distribution $W(\{Q_i,P_i\})$ on the lattice. There is $\partial W/\partial t=LW$, $L=\sum_i L_i$, and L_i is given by

$$L_iW=-\frac{\partial}{\partial Q_i}[WH_{P_i}]+\frac{\partial}{\partial P_i}[W(H_{Q_i}+\Delta_{\epsilon,\eta}^2H_{P_i})] +T\frac{\partial}{\partial P_i}\left[\Delta_{\epsilon,\eta}^2\frac{\partial W}{\partial P_i}\right] . \tag{9}$$

The discrete version of L_i is rather unwieldy, e.g., the drift term for constant ϵ and η becomes

$$\begin{aligned}\partial_i\Delta_{\epsilon,\eta}^2\partial_i=&(\epsilon^2+4\tilde{\eta}\epsilon+6\tilde{\eta}^2)\partial_{i,i}^2\\ &-2(\tilde{\eta}\epsilon+2\tilde{\eta}^2)(\partial_{i,i-2}^2+\partial_{i,i+2}^2)\\ &+\tilde{\eta}^2(\partial_{i,i-4}^2+\partial_{i,i+4}^2) ,\end{aligned}$$

where $\partial_i=\partial/\partial P_i$ and $\partial_{i,j}^2=\partial^2/\partial P_i\partial P_j$ for brevity. It is an easy exercise to show that $Le^{-H/T}=0$ so that thermal equilibrium exists and is the stationary state of the lattice Fokker-Planck equation. Moreover, the structure of Eqs. (5), (6), and (9) suggests that this property holds for an arbitrary dissipation operator $\Delta_{\epsilon,\eta}(\nabla)=\Delta_{\epsilon,\eta}(-\nabla)$ (this condition precludes the introduction of an "arrow of space" akin to the "arrow of time") which can accommodate an arbitrary complex dispersion relation $\omega(k)$. Thus, for example, preferential attenuation of waves with small k vector is obtained if $\Delta_{\epsilon,\eta}\rightarrow\Delta_{\epsilon,\eta}^{-1/2}$ while obviously this term cannot be derived from any finite interaction Hamiltonian of the form (1). Analogous situation obtains also for multiplicative scalar noise provided that it couples to but one of the system variables. Let, e.g., $Q\rightarrow\chi(Q)$ in the interaction Hamiltonian $H_{\rm int}$ of Eq. (1). The stochastic equations of motion (5) and (6) are replaced in this case by the set

$$\dot{P}=-H_Q-\chi_Q\{\Delta_{\epsilon,\eta}^2[\chi_Q\dot{Q}]+(2T)^{1/2}\Delta_{\epsilon,\eta}\dot{w}\} , \tag{10}$$

$$\dot{Q}=H_P , \tag{11}$$

and the Fokker-Planck operator L_i becomes

$$L_iW=-\frac{\partial}{\partial Q_i}\{WH_{P_i}\}+\frac{\partial}{\partial P_i}\{W[H_{Q_i}+\chi_{Q_i}(\Pi_iH)] +T\chi_{Q_i}(\Pi_iW)\} . \tag{12}$$

The generalized dissipation operator Π_i contains also information about the multiplicative action of the noise field:

$$\Pi_i = \Delta^2_{\epsilon,\eta}\chi_{Q_i}\frac{\partial}{\partial P_i} . \tag{13}$$

Note that Eq. (10) without noise can be written in the compact form $\dot{P} = -H_Q - \chi_Q \Pi H$, $\chi_Q = \partial\chi/\partial Q$, which is well defined also in the continuum limit. Both the function $\chi(Q)$ and the operator $\Delta^2_{\epsilon,\eta}$ may be chosen at will. It is therefore possible to construct a stochastic lattice field theory which goes over to thermal equilibrium given the dispersion relation $\omega(k)$ and the function χ. The deterministic damped equations of motion are then defined also in the continuum limit. This construction constitutes our main result, its further developments and limitations are considered below.

The stochastic equation studied so far were explicitly solved for the pairs $\{\dot{Q}_i, \dot{P}_i\}$ and it was easy to write for them a Fokker-Planck equation and to verify that $Le^{-H/T} = 0$. Complications arise, though, if the multiplicative function $\chi \to \chi(Q,P)$ in Eq. (1) since in this case the equations of motion become

$$\dot{P} = -H_Q - \chi_Q[\Delta^2_{\epsilon,\eta}\dot{\chi} + (2T)^{1/2}\Delta_{\epsilon,\eta}\dot{w}] , \tag{14}$$

$$\dot{Q} = H_P + \chi_P[\Delta^2_{\epsilon,\eta}\dot{\chi} + (2T)^{1/2}\Delta_{\epsilon,\eta}\dot{w}] , \tag{15}$$

where $\dot{\chi} = \chi_P\dot{P} + \chi_Q\dot{Q}$. The linear system for $\{\dot{Q}_i, \dot{P}_i\}$ is trivially solved if $\Delta_{\epsilon,\eta} \equiv \epsilon$. This corresponds to two degrees of freedom and the discriminant of the system turns out to be unity. In any other case the system (14),(15) is either infinite or boundary conditions must be assigned on the lattice. Multiplicative noise of this type arises naturally in magnetization studies and we defer an analysis of this problem to the discussion of Eqs. (19) and (20) below.

In order to apply the Hamiltonian formalism to magnetization we first need to parametrize [3] the three components M_i [$|\mathbf{M}(t,\mathbf{r})|$ is conserved and has no spatial dispersion by assumption] in terms of the canonical variables (ϕ, P):

$$\begin{aligned}\mathcal{M} &= \gamma_0 \mathbf{S} \\ &= \gamma_0((P_0^2 - P^2)^{1/2}\cos\phi, (P_0^2 - P^2)^{1/2}\sin\phi, P) .\end{aligned} \tag{16}$$

$\mathcal{M}$ is the magnetic moment residing in some volume v (e.g., a unit cell) with $\mathcal{M} = v\mathbf{M}$ and $\mathbf{S}$ is the corresponding angular momentum. By assumption $|\mathbf{S}| = P_0$ is independent of time and position. It has been shown previously [3] that bilinear coupling of the vector $\mathbf{S}$ to a bath construed as an electromagnetic field, $H_{\text{int}} = \sum_\alpha c_c(\mathbf{S}\cdot\mathbf{B}_\alpha)$, leads to Gilbert's equation of motion discussed in introduction. The magnetic-field vector $\mathbf{B}_\alpha$ is expressed in terms of the normal field modes $(\mathbf{q}_\alpha, \mathbf{p}_\alpha)$ and in the spirit of Eq. (1) we formally generalize this interaction Hamiltonian as

$$H_{\text{int}} = \sum_\alpha \left\{ c_\alpha(\mathbf{S}\cdot\mathbf{B}_\alpha) + \tilde{d}_\alpha \left[\frac{\partial \mathbf{S}}{\partial x}\cdot\frac{\partial \mathbf{B}_\alpha}{\partial x} + \frac{\partial \mathbf{S}}{\partial y}\cdot\frac{\partial \mathbf{B}_\alpha}{\partial y} + \frac{\partial \mathbf{S}}{\partial z}\cdot\frac{\partial \mathbf{B}_\alpha}{\partial z} \right] \right\} . \tag{17}$$

The form of H_{int} determines the operator $\Delta_{\epsilon,\eta}$ and for our purposes any H_{int} with correct symmetry will do; see Eqs. (13) and (22). In the Markovian limit we obtain the stochastic equation for $\mathbf{S}$ by calculating $\partial H_{\text{tot}}/\partial\mathbf{S}$,

$$\mathbf{S}_t = \mathbf{S} \times [-H_\mathbf{S} - \Delta^2_{\epsilon,\eta}\mathbf{S}_t - (2T)^{1/2}\Delta_{\epsilon,\eta}\mathbf{w}_t] . \tag{18}$$

$\Delta_{\epsilon,\eta}$ is given in this special case by Eq. (7), further $H_\mathbf{S} = \partial H/\partial\mathbf{S}$, $\mathbf{S}_t = d\mathbf{S}/dt$, etc. This equation conserves local magnetization $|\mathbf{M}(t,\mathbf{r})|$ and is distinguished by the presence of terms proportional to $\epsilon\eta$ (i.e., mixing homogeneous and inhomogeneous dissipation) absent in Bar'yakhtar's equation [4–6].

There remains yet to establish a Fokker-Planck equation for magnetization distribution and to show that thermal equilibrium is its stationary state. To this end we write down the stochastic equations of motion in terms of (ϕ, P) symbolically as

$$\dot{\phi} = H_P + \mathbf{S}_P\cdot[\Delta^2_{\epsilon,\eta}\mathbf{S}_t + (2T)^{1/2}\Delta_{\epsilon,\eta}\mathbf{w}_t] , \tag{19}$$

$$\dot{P} = -H_\phi - \mathbf{S}_\phi\cdot[\Delta^2_{\epsilon,\eta}\mathbf{S}_t + (2T)^{1/2}\Delta_{\epsilon,\eta}\mathbf{w}_t] , \tag{20}$$

where $\mathbf{S}_P \to \partial S_i/\partial P_i$, etc., and $\mathbf{S}_t = \mathbf{S}_P\dot{P} + \mathbf{S}_\phi\dot{\phi}$. For the Hamiltonian density

$$\hat{H} = E(p^2 + \phi^2)/2 + \beta[(\nabla p)^2 + (\nabla\phi)^2]/2 ,$$

with $p = P/P_0$, there follows a dispersion relation of the form (8) where $\omega_1^2 = P_0^{-2}(E + \beta k^2)^2(1 + K^2)^{-1}$ and $\omega_2 = K\omega_1$. The quantity $K = P_0(\epsilon + \eta k^2)^2$ represents an effective dissipation constant. For large k both ω_1 and ω_2 go to zero. This is a peculiarity of Gilbert's equation in which an overdamped system behaves like an underdamped system with a very large inertia [2]. The underdamped limit condition $K \ll 1$ is satisfied for small k only and it is only in this limit that we could treat the above stochastic system consistently.

The linear systems (14),(15) and (19),(20) have similar structure and cannot, in general, be solved for the conjugate variables pair. The problem, however, is not merely one of a large number of linear equations or of boundary conditions: It was shown [3] that stochastic systems of this form go to thermal equilibrium if and only if the discriminant $D(\phi, P)$ of the linear system is independent of the canonical variables, that is, if $\dot{D}(\phi, P) = 0$ on the whole phase space. This condition must be satisfied in addition to the usual FD theorem. There is no guarantee therefore that the system (19),(20) evolves to thermal equilibrium, in fact, a three-site calculation with periodic boundary conditions suggests that it does not. This simple result constitutes by no means a proof but it seems highly unlikely that a consistent stochastic theory of the form (18) or (19),(20) exists at all. Violations of the discriminant condition may, however, be neglected in the first order of a small dissipation constant [3]. We iterate Eqs. (19) and (20) by letting $\mathbf{S}_t = H_P\mathbf{S}_\phi - H_\phi\mathbf{S}_P$ and for the *underdamped* system obtain a (Stratonovich) Fokker-

Planck equation of the form $\partial W/\partial t = \sum_i L_i W$, where

$$L_i W = -\frac{\partial}{\partial \phi_i}\left\{ W\left[H_{P_i} + \frac{\partial \mathbf{S}_i}{\partial P_i}\cdot(\mathbf{\Pi}_i H)\right] + T\frac{\partial \mathbf{S}_i}{\partial P_i}\cdot(\mathbf{\Pi}_i W)\right\} + \frac{\partial}{\partial P_i}\left\{ W\left[H_{\phi_i} + \frac{\partial \mathbf{S}_i}{\partial \phi_i}\cdot(\mathbf{\Pi}_i H)\right] + T\frac{\partial \mathbf{S}_i}{\partial \phi_i}\cdot(\mathbf{\Pi}_i W)\right\} . \quad (21)$$

The operator $\mathbf{\Pi}_i$ is in this case defined as

$$\mathbf{\Pi}_i = \Delta^2_{\epsilon,\eta}\left[\frac{\partial \mathbf{S}_i}{\partial \phi_i}\frac{\partial}{\partial P_i} - \frac{\partial \mathbf{S}_i}{\partial P_i}\frac{\partial}{\partial \phi_i}\right] , \quad (22)$$

and $Le^{-H/T}=0$ as desired. The underdamped noiseless equations (19) and (20) have again the compact form $\dot{\phi}=H_P+(\mathbf{S}_P\cdot\mathbf{\Pi})H$ and $\dot{\mathbf{P}}=-H_\phi-(\mathbf{S}_\phi\cdot\mathbf{\Pi})H$ and are well defined in the continuum limit. For $\Delta_{\epsilon,\eta}=\epsilon$ these equations become the underdamped Gilbert's equation [2,3]. The multiplicative noise driving the system (18) determines the phase-space action of the dissipative operator $\mathbf{\Pi}$ but, as in Eq. (13), the operator $\Delta_{\epsilon,\eta}$ may be chosen at will. Therefore, given a small k dispersion relation for spin waves (large wave vectors are excluded *a priori* in this underdamped limit, $K \ll 1$), one can construct a suitable operator $\Delta_{\epsilon,\eta}$ from the linearized (noiseless) equations (19) and (20). Obviously, dissipative couplings of more complex symmetry [3] than $H_{\rm int}$ of Eq. (17) may be considered; they redefine the phase-space action of the operator $\mathbf{\Pi}$ similar to the choice of χ in Eq. (13).

To summarize, the stationary state of the Fokker-Planck equation corresponding to a scalar field driven either by additive noise or by multiplicative noise coupled to but one canonical variable is thermal equilibrium. Within the framework of stochastic equations the dissipative coupling may be arbitrary. In stochastic theory of scalar or vector fields driven by more complicated multiplicative noise thermal equilibrium exists in general in the underdamped limit only. We next treat dissipative dynamics of magnetization (two-dimensional field driven by multiplicative noise), propose an alternative equation of motion and show that it evolves to thermal equilibrium in the underdamped limit. A possible deterministic form is

$$\frac{d\mathbf{M}}{dt}=\gamma_0\mathbf{M}\times\left[-\frac{\delta H}{\delta \mathbf{M}}-(\epsilon_g^{1/2}-\eta_g\nabla^2)^2\frac{d\mathbf{M}}{dt}\right] \quad (23)$$

since the dissipation operator (7) yields a quadratic dispersion relation for spin waves, as it should. However, this is obviously not the only possible choice and the relation between the dissipation operator $\Delta_{\epsilon,\eta}$ and domain-wall dynamics (see, e.g., Refs. [4,10]) is the subject of current research.

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